Particular Solutions for a (3+1)-dimensional Generalized Shallow Water Wave Equation

Yi-Tian Gao[a,b], Bo Tian[a], and Woopyo Hong[b]

[a] Depts. of Applied Physics and Mathematics, Beijing University of Aeronautics and Astronautics, Beijing 100083, China*
[b] Dept. of Physics, Catholic University of Taegu-Hyosung, Hayang, Kyongsan, Kyungbuk 712-702, South Korea

Z. Naturforsch. **53a,** 806–807 (1998); received May 8, 1998

The shallow water wave equations (SWWEs) are of current interest in nonlinear sciences. In this paper we obtain a new family of soliton-like solutions for a (3-1)-dimensional generalized SWWE. Samples are given.

The shallow water wave equations (SWWEs) are of current interest in many fields of non-linear sciences. However, so far only the (1+1)-dimensional SWWE has been discussed in details (see, e.g., [1] and references therein). Its (3+1)-dimensional generalization [2],

$$u_{yt} + u_{xxxy} - 3\, u_{xx}\, u_y - 3\, u_x\, u_{xy} - u_{xz} = 0\,, \qquad (1)$$

is known not to be completely integrable in the usual sense, though originated as the second equation in the Kadomtsev-Petviashvili hierarchy [2, 3]. Four families of solutions for [1] have been found recently [4].

In this paper we investigate (1) with a direct method [5, 6, 7].

To begin with, we insert a particular ansatz, namely

$$u\,(x, y, z, t) = A\, \partial_x^m \partial_y^n\, w\,[q\,(x, y, z, t)] + B\,, \qquad (2)$$

into (1), where $A \neq 0$ and B are constants, while $m = 1$ and $n = 0$ are the integers determined via the leading-order analysis.

After steps of computerized symbolic computation, we split (1) by respectively equating to zero the coefficients of the terms with the highest power of the differential coefficients of $q\,(x, y, z, t)$ and the coefficients of the w' terms, i.e.,

$$q_x^4\, q_y\, [-6\, A\, w''\, w''' + w^{(5)}] = 0\,, \qquad (3)$$

$$q_{xyt} - q_{xxz} + q_{4x,y} = 0\,. \qquad (4)$$

Equation (3) is a fifth-order ordinary differential equation, the general solution of which has five constants of integration. Similarly, a partial differential equation, like (4), might have its general solution with certain arbitrary functions. Hereby, for simplicity and for the solitonic features, we select the set of particular solutions of the x-linear form

$$w\left[q\,(x, y, z, t)\right] = -\frac{2}{A} \ln\left[q\,(x, y, z, t)\right], \qquad (5)$$

$$q\,(x, y, z, t) = 1 + e^{ax + \Psi(y,z,t)}\,, \qquad (6)$$

where $a \neq 0$ is a constant and $\Psi\,(y, z, t)$ is a differentiable function. The set does not include constants of integration but one arbitrary function.

Again we substitute (5) and (6) back into (1) and equate to zero the coefficients of like powers of $\exp\,[a\,x + \Psi\,(y, z, t)]$, to get the couple of constraints

$$\Psi_{yt} = 0\,, \qquad (7)$$

$$a\, \Psi_z - (a^3 + \Psi_t)\, \Psi_y = 0\,. \qquad (8)$$

Integrating (7) leads to its general solution

$$\Psi\,(y, z, t) = \phi\,(y, z) + \lambda\,(z, t)\,, \qquad (9)$$

which is then substituted back into (8) to get

$$a\,(\phi_z + \lambda_z) - \phi_y\,(a^3 + \lambda_t) = 0\,. \qquad (10)$$

This way we obtain the family of soliton-like solutions for (1)

$$u\,(x, y, z, t) = B - \frac{2\,a\,e^{ax+\phi(y,z)+\lambda(z,t)}}{1 + e^{ax+\phi(y,z)+\lambda(z,t)}} = F - a \cdot \tanh\left[\frac{a\,x + \phi(y,z) + \lambda(z,t)}{2}\right], \qquad (11)$$

where a, $\lambda\,(z, t)$ and $\phi\,(y, z)$ must satisfy the constraint (10), while $F = B - a$ remains an arbitrary constant.

The family is different from those in [4] but does exist, from which we present some examples:

Sample 1: Consider the assumptions

$$\phi\,(y, z) = b\,y + \omega\,(z) \text{ and } \lambda\,(z, t) = e^{\alpha\, t}\, \delta\,(z)\,, \qquad (12)$$

where b and α are constants while $\omega\,(z)$ and $\delta\,(z)$ are differentiable functions. Equation (10) reduces to the set of equations

$$\begin{cases} a\,\delta_z - b\,\alpha\,\delta = 0\,, \\ \omega_z - b\,a^2 = 0\,. \end{cases} \qquad (13)$$

Reprint request to Prof. Bo Tian;
* Mailing address.

0932-0784 / 98 / 0900-0806 $ 06.00 © – Verlag der Zeitschrift für Naturforschung, Tübingen · www.znaturforsch.com

Integration of them over y leads to

$$\begin{cases} \delta(z) = \beta\, e^{b\alpha z/a}, \\ \omega(z) = b\, a^2 z + \gamma, \end{cases} \tag{14}$$

where β and γ are constants. We thus obtain the result

$$u(x,y,z,t) = F - a \cdot \tanh \cdot \left[\frac{\beta\, e^{\alpha(t+bz/a)} + a x + b y + a^2 b z + \gamma}{2} \right]. \tag{15}$$

Sample 2: Solitary waves are a special case of the family, since we are able to assume that

$$\phi(y, z) + \lambda(z, t) = b\, y + c\, z + p\, t + h, \tag{16}$$

where b, c, p and h are constants. Constraint (10) then leads to

$$u(x,y,z,t) = F - a \cdot \tanh \cdot \left[\frac{a x + b y + c z + a\left(\frac{c}{b} - a^2\right) t + h}{2} \right]. \tag{17}$$

Acknowledgements

We thank Prof. Q. S. Lu and the referee for their valuable comments. YTG and BT were supported by the Fund for Excellent Young University Teachers and the Research Grants for Scholars Returning from Abroad, State Education Commission of China; and supported by the National Natural Science Foundation of China. WPH was supported by Catholic University of Taegu-Hyosung.

[1] P. Clarkson and E. Mansfield, Nonlinearity, **7,** 975 (1994).
[2] M. Jimbo and T. Miwa, Publ. Res. Inst. Math. Sci., Kyoto Univ. **19,** 943 (1983).
[3] B. Dorizzi, B. Gammaticos, A. Ramani, and P. Winternitz, J. Math. Phys. **27,** 2848 (1986).
[4] B. Tian and Y. T. Gao, Z. Naturforsch. **51 a,** 171 (1996).
[5] Y. T. Gao and B. Tian, Computers Math. Applic. **30,** 97 (1995).
[6] B. Tian and Y. T. Gao, J. Phys. A **29,** 2895 (1996).
[7] W.-H. Steeb, Continuous Symmetrics, Lie Algebras, Differential Equations and Computer Algebra, World Scientific, Singapore 1996.

New Exact Solutions for a Generalized Breaking Soliton Equation

Yi-Tian Gao[a,b], Bo Tian[a], and Woopyo Hong[b]

[a] Depts. of Applied Physics and Mathematics, Beijing University of Aeronautics and Astronautics, Beijing 100083, China*
[b] Dept. of Physics, Catholic University of Taegu-Hyosung, Hayang, Kyongsan, Kyungbuk 712-702, South Korea

Z. Naturforsch. **53 a,** 807–808 (1998); received May 8, 1998

The breaking soliton equations are a class of nonlinear evolution equations of broad interest in physical and mathematical sciences. In this paper, the application of the generalized tanh method with symbolic computation leads to new exact solutions for a generalized breaking soliton equation, of which the previously-obtained solutions are the special cases.

Within a decade, a class of nonlinear evolution equations, called the breaking soliton equations, has become a widely interesting subject in physical and mathematical sciences, as seen, e.g., in [1–5]. In addition to the rich mathematical properties, those equations are found to include the self-dual Yang-Mills equation, and to be of value in describing the (2+1)-dimensional interaction of the Riemann waves and long waves. A generalized breaking soliton equation [1, 2] reads as

$$(u_{xt} - 4\, u_x\, u_{xy} - 2\, u_y\, u_{xx} + u_{xxxy})_x = -\alpha^2\, u_{yyy}, \tag{1}$$

where α^2 is real. Two classes of exact solutions of (1) have been found, one of which is the solitary waves plus arbitrary functions of t [4], the other is linear with respect to y but independent of α^2 [5].

In this paper, we apply the computerized symbolic computation and generalized tanh method [6] to (1), assuming that the exact solutions are of the form

$$u(x, y, t) = \sum_{m=0}^{N} A_m(y, t) \cdot \tanh^m [\Psi(y, t)\, x + \Theta(y, t)], \tag{2}$$

Reprint request to Prof. Bo Tian;
* Mailing address.